MATC44 Week 6 Notes

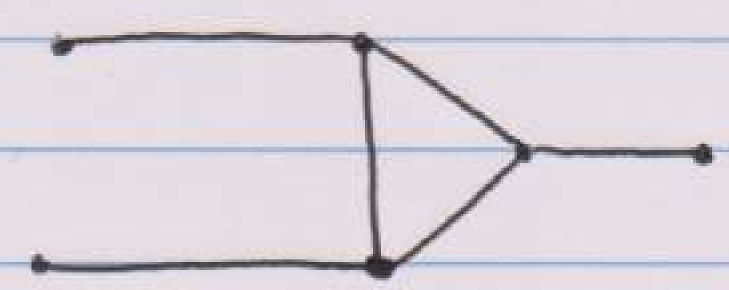
1. Euler's Characteristic:

a) Planar Graph:

- Euler's characteristic states that if a finite, connected, planar graph is drawn in the plane without any edge intersections, and v = num of vertices and e = num of edges and f = num of faces (regions bounded by edges including the outer, infinitely large region) then: $v - e + f = 2$.

- E.g.

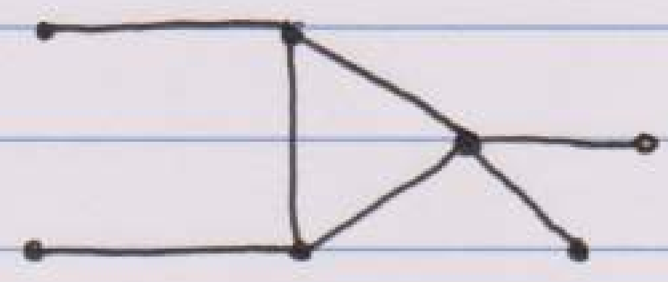
1.



V	6
E	6
F	2

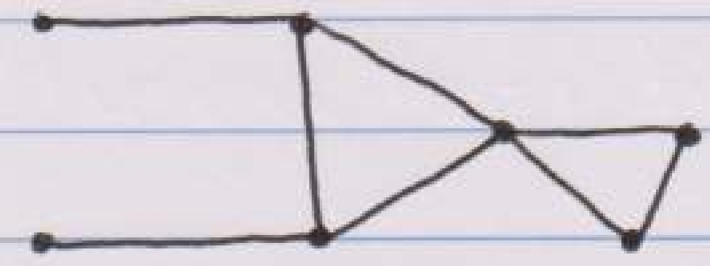
Note: The 2 faces are the one in the triangle and the area/region outside.

2.



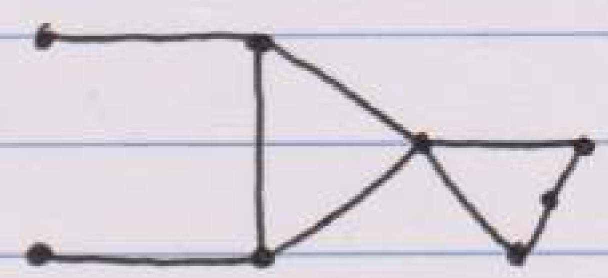
V	7
E	7
F	2

3.



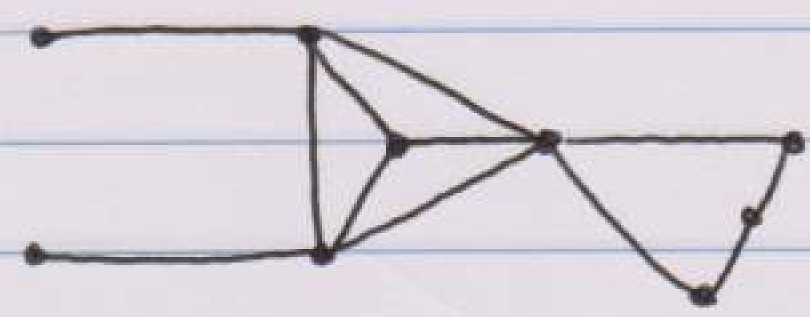
V	7
E	8
F	3

4.



V	8
E	9
F	3

5.



V	9
E	12
F	5

	1	2	3	4	5
V	6	7	7	8	9
E	6	7	8	9	12
F	2	2	3	3	5
ΔV	–	+1	+0	+1	+1
ΔE	–	+1	+1	+1	+3
ΔF	–	+0	+1	+0	+2

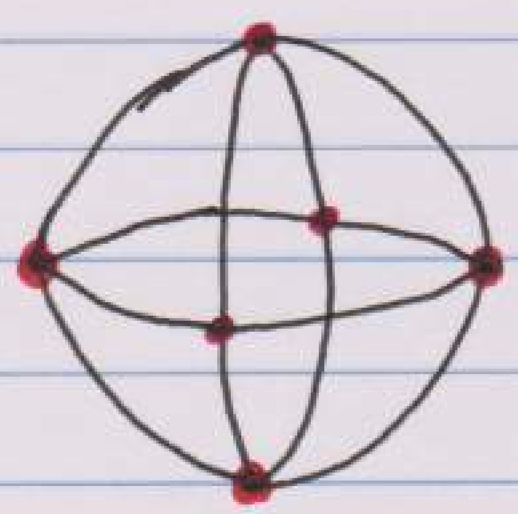
- Thm:
  $\Delta V + \Delta F = \Delta E \leftrightarrow \Delta V - \Delta E + \Delta F = 0$
  $\leftrightarrow \Delta(V-E+F) = 0$
  $\leftrightarrow V-E+F$ is an invariant

  For planar graphs, $V-E+F=2$.

b) Spheres:

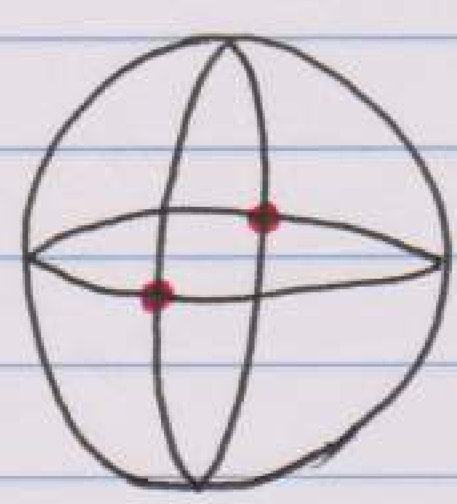
- Euler characteristic formula for a sphere is $V-E+F=2$.
  Note: The outer region is not included in F for spheres.

- E.g.



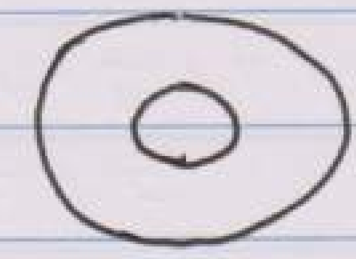
$$\left.\begin{array}{l} V = 6 \\ E = 12 \\ F = 8 \end{array}\right\} \begin{array}{l} V-E+F \\ = 6-12+8 \\ = 2 \end{array}$$

- E.g.



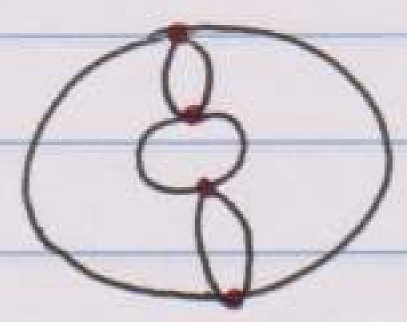
$$\left.\begin{array}{l} V = 2 \\ E = 4 \\ F = 4 \end{array}\right\} \begin{array}{l} V-E+F \\ = 2-4+4 \\ = 2 \end{array}$$

c) Torus

- Looks like this: 

- The Euler Characteristic for a torus is $V-E+F=0$.

- E.g.



$$\left.\begin{aligned} V &= 4 \\ E &= 8 \\ F &= 4 \end{aligned}\right\} \begin{aligned} &V-E+F \\ &= 4-8+4 \\ &= 0 \end{aligned}$$

- The Euler Characteristic for any number of holes generalizes to $V-E+F = 2-2N$ where N is the number of holes.

2. Principles of Counting

a) Additive Principle:

- Version 1: If process A can be performed in n ways and if process B can be performed in m ways, then either A OR B can be performed in $n+m$ ways, for independent processes A, B.

- Version 2: If A and B are sets s.t. $A \cap B = \varnothing$, then $|A \cup B| = |A| + |B|$.

Note: $|A|$ means the number of elements in set A.

b) Multiplicative Principle:

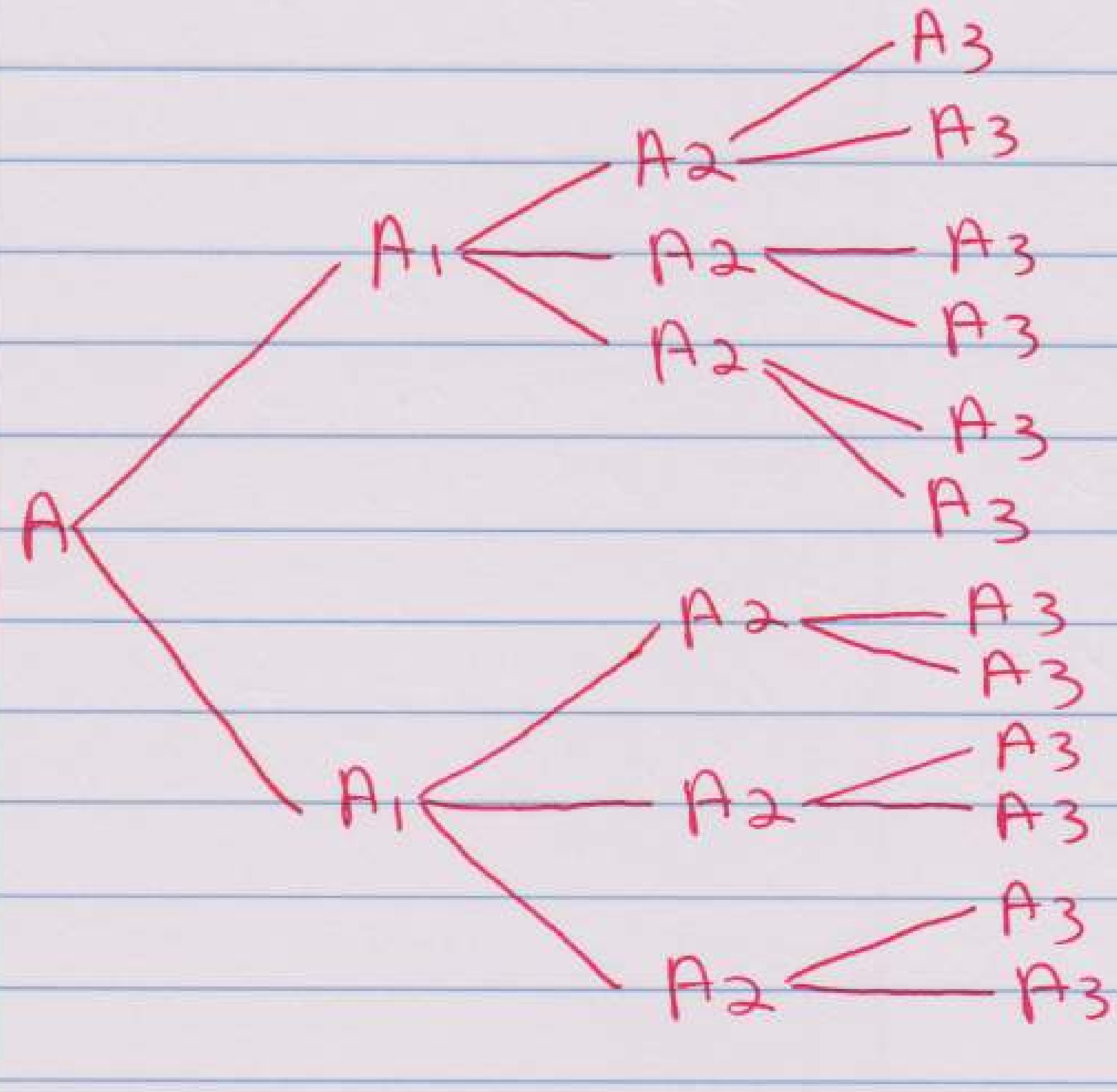
- Version 1: If process A can be completed in n ways and if process B can be performed in m ways, then A AND B can be performed in $n \cdot m$ ways.

- Version 2: $A \times B$ is the Cartesian Product of A and B. $A \times B = \{(a,b) \mid a \in A, b \in B\}$. I.e. $A \times B$ is the set of all ordered pairs (a,b) s.t. a is in A and b is in B. If A and B are sets, then $|A \times B| = |A| \cdot |B|$

- Version 3: Consider process A which is a composition of other, simpler processes $A_1, A_2, \dots A_k$, such that you must perform A_1 first, followed by A_2, followed by $A_3, \dots,$ and A_k is the last process performed.

If there are n_1 ways to perform A_1, n_2 ways to perform $A_2, \ldots, n_k$ ways to perform A_k, then the process A can be performed in $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ ways.

Remark: To visualize this, consider the following: Suppose that in order to perform process A you must perform process A_1 followed by process A_2 followed by process A_3. Suppose that there are 2 ways to perform A_1, 3 ways to perform A_2 and 2 ways to perform A_3. Then, process A can be performed in $2 \cdot 3 \cdot 2$ or 12 ways.

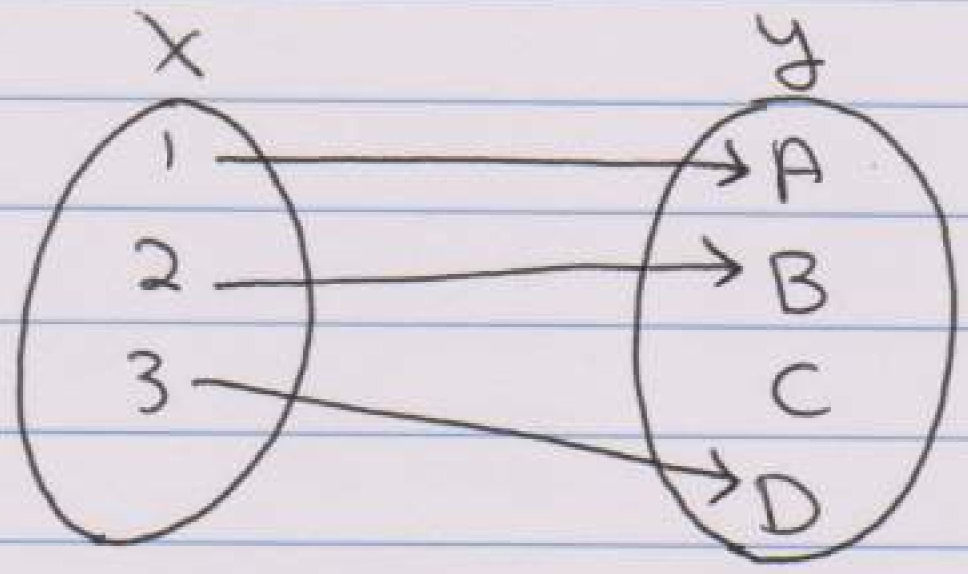


c) Bijection Principle:

- Let A and B be 2 sets. A function $f: A \to B$ is called

  a) injective if for all $x_1 \neq x_2$ in A, we have $f(x_1) \neq f(x_2)$ in B. I.e. A function is injective if it maps distinct elements of its domain to distinct elements of its codomain.

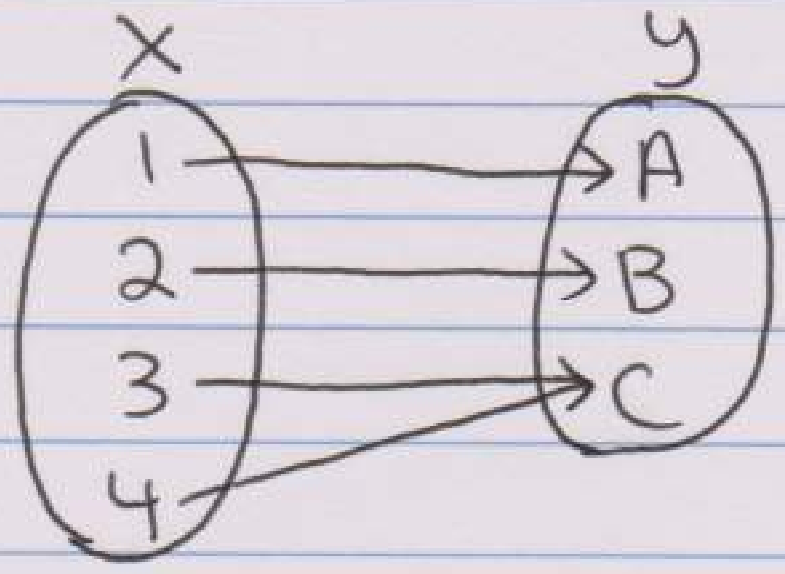
  E.g.



This is an injective function.

  b) surjective if for all $y \in B$ there is an $x \in A$ s.t. $f(x) = y$. I.e. A function is surjective if for every element y in the codomain B of f there is at least one element x in the domain A of f s.t. $f(x) = y$. It is not required that x be unique. The function f may map 1 or more elements of A to the same element of B.

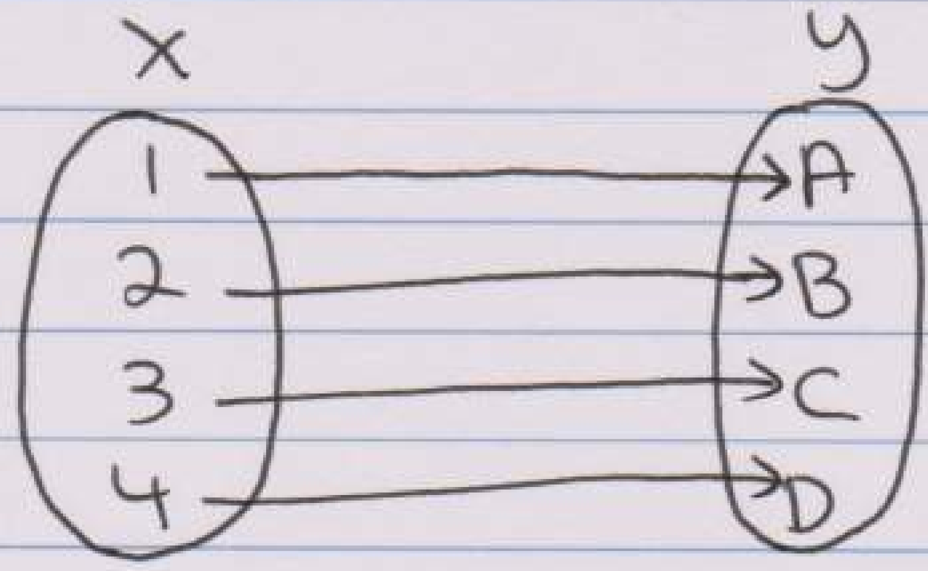
E.g.



This is a surjective function.

c) bijective if it is both injective and surjective. I.e. A function is bijective if it maps each element of one set with exactly 1 element of another set and each element of the second set is mapped with exactly 1 element of the first set.
There are no unmapped elements.

E.g.



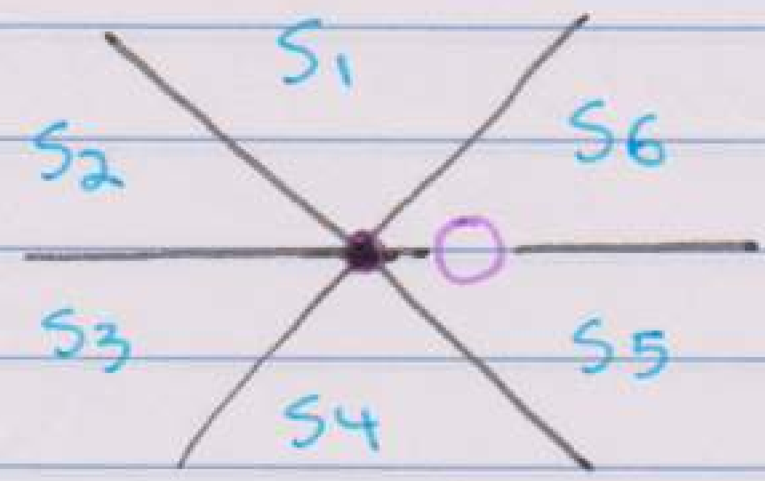
A bijective function.

Principle of Bijections: Two finite sets, A, B, have the same number of elements if there is a bijection from A to B.

d) Examples:

1. Consider 3 lines on the plane passing through the point O creating 6 sectors in total. We put 5 points in each sector, so there are 30 points in total. Show that there are at least 1000 triangles with vertices from 3 of the 30 points s.t. the point O is either in the interior or on the edge of these triangles.

Soln:



Case 1: Consider any 3 points s.t. 1 is from S_1, 1 is from S_3 and 1 is from S_5. If we form a triangle with those 3 vertices, point O must be contained in it.

Similarly, if we consider any 3 points s.t. 1 is from S_2, 1 is from S_4 and 1 is from S_6, then any triangle created by those 3 points must contain O.

Since there are 5 points in S_1, 5 points in S_3 and 5 points in S_5, we can create $5 \cdot 5 \cdot 5$ or 125 triangles from vertices in S_1, S_3 and S_5. By the exact logic/math, we can create 125 triangles from vertices in S_2, S_4 and S_6. Now, we have $125 + 125$ or 250 triangles.

Case 2: Consider a pair of points from opposite sectors (S_1, S_4), (S_2, S_5), (S_3, S_6). Consider the case where the points are taken from $(1, 4)$. Consider the line created by these 2 points. The point O must either lie to the left of the line or to the right. If O lies to the left, then if we choose any point in either S_2 or S_3, we can create a triangle that contains O. Similarly, if O lies right of the line, if we choose any point from S_5 or S_6, we can create a triangle that contains O.

There are 5 points in each sector, so for the case with (1,4), there are $5 \cdot 5 \cdot (5+5)$ or 250 triangles. The $5+5$ is from choosing a point in either S_2 or S_3. However, there are 3 pairs of opposite sectors, so in total, there are $3 \cdot 250$ or 750 triangles.

Putting everything together, we have $250 + 750$ or 1000 triangles.

2. Compute the number of squares on the plane with vertices of the form (x,y) with $x, y \in \mathbb{N}$ s.t. $1 \le x \le n$, $1 \le y \le n$.

Soln:

Note: Both of the following are considered valid squares for this question:

a) b)

We will find the number of squares for each type separately and combine the count for the answer. Let's start with a.

Consider the bottom left vertex, (a,b), of each square.

- If there are unit squares, then there are $(n-1)^2$ of these points and hence $(n-1)^2$ unit squares.

- If there are 2×2 squares, there are $(n-2)^2$ of these points and hence $(n-2)^2$ 2 by 2 squares.

- If there are $(n-1) \times (n-1)$ squares, there is only 1 of these points and hence 1 $(n-1) \times (n-1)$ square.

In general, for a $k \times k$ square, there are $(n-k)^2$ of these points and hence $(n-k)^2$ of these $k \times k$ squares where $1 \leq k < n$. Note: If there are n vertices, then there are $n-1$ edges connecting these n vertices in a line.

In total, there are

$\sum_{k=1}^{n-1} (n-k)^2$ of these upright squares.

Now, let's consider the slanted squares. Given any $k \times k$ upright square, we can find $k-1$ inscribed slanted square. Thus, there are

$$\sum_{k=1}^{n-1} (n-k)^2 (k-1) \text{ slanted squares.}$$

Putting it all together, we have

$$\sum_{k=1}^{n-1} (n-k)^2 + \sum_{k=1}^{n-1} (n-k)^2 (k-1)$$

$$= \sum_{k=1}^{n-1} (n-k)^2 + (n-k)^2 \cdot k - (n-k)^2$$

$$= \sum_{k=1}^{n-1} k(n-k)^2$$

$$= \sum_{k=1}^{n-1} k(n^2 - 2nk + k^2)$$

$$= \sum_{k=1}^{n-1} kn^2 - 2nk^2 + k^3$$

$$= n^2 \sum_{k=1}^{n-1} k - 2n \sum_{k=1}^{n-1} k^2 + \sum_{k=1}^{n-1} k^3$$

Note: Consider the previous sums

$$\sum_{k=1}^{n-1} (n-k)^2 \quad \text{and} \quad \sum_{k=1}^{n-1} (n-k)^2 (k-1)$$

If we changed the sums s.t. it goes to n instead of n-1, there would be no difference.

I.e.

$$\sum_{k=1}^{n} (n-k)^2 = \sum_{k=1}^{n-1} (n-k)^2 \quad \text{and}$$

$$\sum_{k=1}^{n} (n-k)^2 (k-1) = \sum_{k=1}^{n-1} (n-k)^2 (k-1)$$

This is because if $k=n$, $(n-k)^2 = 0$. As such, we can replace $n-1$ with n. Continuing from before, we get

$$n^2 \sum_{k=1}^{n} k - 2n \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k^3$$

Since we know that:

1. $\sum_{k=1}^{n} k = \frac{(n)(n+1)}{2}$

2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{k=1}^{n} k^3 = \frac{(n^2)(n+1)^2}{4}$

we can replace our previous sums with these. When we do, we get the following:

$$\frac{(n^2)(n)(n+1)}{2} - \frac{(2n)(n)(n+1)(2n+1)}{6} + \frac{(n)^2(n+1)^2}{4}$$

$$= \frac{(n^3)(n+1)}{2} - \frac{(n^2)(n+1)(2n+1)}{3} + \frac{(n^2)(n+1)^2}{4}$$

$$= \frac{6n^3(n+1) - 4n^2(n+1)(2n+1) + 3n^2(n+1)^2}{12}$$

$$= \left(\frac{n^2(n+1)}{12}\right)\left(6n - 4(2n+1) + 3(n+1)\right)$$

$$= \left(\frac{n^2(n+1)}{12}\right)\left(6n - 8n - 4 + 3n + 3\right)$$

$$= \left(\frac{n^2(n+1)}{12}\right)\left(n-1\right)$$

$$= \frac{(n^2)(n+1)(n-1)}{12}$$